Suggested Solution to Problem Set 1[1]

1 Integration

(i) First, we write $x^2 + 2x + 4 = (x+1)^2 + 3$.

Letting $x + 1 = \sqrt{3}\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then, $dx = \sqrt{3}\sec^2\theta\, d\theta$.

The integral becomes

$$\begin{aligned}
\int \frac{x}{x^2+2x+4}\,dx &= \int \frac{\sqrt{3}\tan\theta - 1}{3(\tan^2\theta+1)} \cdot \sqrt{3}\sec^2\theta\, d\theta \\
&= \frac{\sqrt{3}}{3}\int \left(\sqrt{3}\tan\theta - 1\right)\, d\theta \\
&= \frac{\left(\sqrt{3}\right)^2}{3}\int \tan\theta\, d\theta - \frac{\sqrt{3}}{3}\int 1\, d\theta \\
&= \ln|\sec\theta| - \frac{\sqrt{3}}{3}\theta + C \\
&= \ln\left|\sqrt{\frac{x^2+2x+4}{3}}\right| - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C \\
&= \frac{1}{2}\ln(x^2+2x+4) - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C'
\end{aligned}$$

where C, C' are arbitrary constants.

(ii) First, we write

$$\frac{3x^2-2x}{1+x^2} = \frac{3(1+x^2)-2x-3}{1+x^2} = 3 - \frac{2x+3}{1+x^2}$$

Then, we have

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{d(1+x^2)}{1+x^2} = \ln(1+x^2) + c_1$$

and

$$\int \frac{3}{1+x^2}\,dx = 3\tan^{-1}x + c_2$$

where c_1, c_2 are arbitrary constants.

Thus, we have

$$\begin{aligned}
\int \frac{3x^2-2x}{1+x^2}\,dx &= \int 3\,dx - \int \frac{2x}{1+x^2}\,dx - \int \frac{3}{1+x^2}\,dx \\
&= 3x - \ln(1+x^2) - 3\tan^{-1}x + C
\end{aligned}$$

where C is an arbitrary constant.

[1] If you have any problems or typos, please contact me via **maxshung.math@gmail.com**

(iii) Since $\dfrac{1}{2\cos 1.5x} = \dfrac{1}{2}\sec\left(\dfrac{3x}{2}\right)$, hence we have

$$\begin{aligned}\int \frac{1}{2\cos 1.5x}\,dx &= \frac{2}{3}\int \frac{1}{2}\sec\left(\frac{3x}{2}\right)\,d\left(\frac{3x}{2}\right)\\ &= \frac{1}{3}\int \frac{\sec\left(\frac{3x}{2}\right)\left[\sec\left(\frac{3x}{2}\right)+\tan\left(\frac{3x}{2}\right)\right]}{\sec\left(\frac{3x}{2}\right)+\tan\left(\frac{3x}{2}\right)}\,d\left(\frac{3x}{2}\right)\end{aligned}$$

Note that

$$d\left[\sec\left(\frac{3x}{2}\right)+\tan\left(\frac{3x}{2}\right)\right] = \left[\sec\left(\frac{3x}{2}\right)\tan\left(\frac{3x}{2}\right)+\sec^2\left(\frac{3x}{2}\right)\right]d\left(\frac{3x}{2}\right)$$

Thus, we have

$$\begin{aligned}\int \frac{1}{2\cos 1.5x}\,dx &= \frac{1}{3}\int \frac{d\left[\sec\left(\frac{3x}{2}\right)+\tan\left(\frac{3x}{2}\right)\right]}{\left[\sec\left(\frac{3x}{2}\right)+\tan\left(\frac{3x}{2}\right)\right]}\\ &= \frac{1}{3}\ln\left|\sec\left(\frac{3x}{2}\right)+\tan\left(\frac{3x}{2}\right)\right| + C\end{aligned}$$

where C is an arbitrary constant.

(iv) (1) Observe that $\dfrac{1}{1+\sin^2 t} = \dfrac{1}{2\sin^2 t+\cos^2 t} = \dfrac{\sec^2 t}{2\tan^2 t+1}$.

Define $f(t) = \dfrac{1}{1+\sin^2 t}$. Note that

$$f(\pi - t) = \frac{1}{1+\sin^2(\pi-t)} = \frac{1}{1+\sin^2 t} = f(t)$$

Therefore, we have

$$\begin{aligned}\int_0^{\pi} f(t)\,dt &= \int_0^{\frac{\pi}{2}} f(t)\,dt + \int_{\frac{\pi}{2}}^{\pi} f(t)\,dt\\ &= \int_0^{\frac{\pi}{2}} f(t)\,dt + \int_0^{\frac{\pi}{2}} \underbrace{f(\pi-t)}_{f(t)}\,dt\\ &= 2\int_0^{\frac{\pi}{2}} f(t)\,dt\end{aligned}$$

Next, we compute the definite integral as follows:

$$\begin{aligned}\int_0^{\pi} \frac{1}{1+\sin^2 t}\,dt &= 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 t}{2\tan^2 t+1}\,dt\\ &= 2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d(\tan t)}{\tan^2 t+\frac{1}{2}}\\ &= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan t\right)\Big|_0^{\frac{\pi}{2}}\\ &= \sqrt{2}\left(\frac{\pi}{2}-0\right)\\ &= \frac{\sqrt{2}\pi}{2}\end{aligned}$$

(2) Again, we need to be careful that the substitution is not a bijection.

Define $f(t) = \dfrac{\cos t}{1+\sin^2 t}$ and observe for $\frac{\pi}{2} \le t \le \pi$, that

$$f(\pi - t) = \frac{\cos(\pi - t)}{1+\sin^2(\pi - t)} = -\frac{\cos t}{1+\sin^2 t} = -f(t)$$

Therefore, we have

$$\begin{aligned}
\int_0^{\pi} f(t)\,dt &= \int_0^{\frac{\pi}{2}} f(t)\,dt + \int_{\frac{\pi}{2}}^{\pi} f(t)\,dt \\
&= \int_0^{\frac{\pi}{2}} f(t)\,dt + \int_0^{\frac{\pi}{2}} \underbrace{f(\pi - t)}_{-f(t)}\,dt \\
&= \int_0^{\frac{\pi}{2}} f(t)\,dt - \int_0^{\frac{\pi}{2}} f(t)\,dt \\
&= 0
\end{aligned}$$

Thus, we have $\displaystyle\int_0^{\pi} \frac{\cos t}{1+\sin^2 t}\,dt = 0$.

(3) Define $g(t) = \dfrac{\sin 2t}{1+\sin^2 t}$.

Since $g(t)$ is not bijective on $[0, \pi]$, so we need to separate it into $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \pi]$.

Observe that $g(\pi - t) = -g(t)$ for any $t \in [\frac{\pi}{2}, \pi]$, so we have

$$\begin{aligned}
\int_0^{\pi} \frac{\sin 2t}{1+\sin^2 t}\,dt &= \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{1+\sin^2 t}\,dt + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin 2t}{1+\sin^2 t}\,dt \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{1+\sin^2 t}\,dt + \int_0^{\frac{\pi}{2}} \frac{\sin(2\pi - 2t)}{1+\sin^2(\pi - t)}\,dt \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{1+\sin^2 t}\,dt - \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{1+\sin^2 t}\,dt \\
&= 0
\end{aligned}$$

Remark. Some of you may do in this way:

$$\int_0^{\pi} \frac{\sin 2t}{1+\sin^2 t}\,dt = \int_0^{\pi} \frac{d(1+\sin^2 t)}{1+\sin^2 t} = \ln(1+\sin^2 t)\Big|_0^{\pi} = 0$$

It is incorrect although the numerical answer is the same as the substitution $u(t) = 1+\sin^2 t$ is not bijective from $[0, \pi]$ to $[1, 2]$.

(v) (Partial fraction decomposition)

(1) Multiplying both sides by $(x-a)(x-b)$ and gives

$$1 \equiv C(x-b) + D(x-a)$$

for any x. Assuming $a - b \neq 0$, we have $\begin{cases} C + D = 0 \\ -bC - Da = 1 \end{cases}$

and hence $C = \dfrac{1}{a-b}$ and $D = \dfrac{-1}{a-b}$.

(2) (a) Note that

$$
\begin{aligned}
\frac{1}{3x^2+10x+3} = \frac{1}{(3x+1)(x+3)} &= \frac{1}{3}\left(\frac{1}{(x-(-1/3))(x-(-3))}\right) \\
&= \frac{1}{3}\left(\frac{1/(3-1/3)}{x+1/3} - \frac{1/(3-1/3)}{x+3}\right) \\
&= \frac{3}{8}\left(\frac{1}{3x+1}\right) - \frac{1}{8}\left(\frac{1}{x+3}\right)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\int_0^1 \frac{1}{3x^2+10x+3}dx = \frac{3}{8}\int_0^1 \frac{1}{3x+1}dx - \frac{1}{8}\int_0^1 \frac{1}{x+3}dx &= \left[\frac{1}{8}\ln|3x+1| - \frac{1}{8}\ln|x+3|\right]_0^1 \\
&= \frac{1}{4}\ln 2 - \frac{1}{4}\ln 2 - 0 + \frac{1}{8}\ln 3 \\
&= \frac{\ln 3}{8}
\end{aligned}
$$

(b) Rewrite $\dfrac{x}{2x^2-4x-6} = \dfrac{x}{2(x+1)(x-3)}$. Then, we observe that

$$
\begin{aligned}
\frac{x}{2(x+1)(x-3)} = \frac{x+1-1}{2(x+1)(x-3)} &= \frac{1}{2(x-3)} - \frac{1}{2(x+1)(x-3)} \\
&= \frac{1}{2(x-3)} - \frac{1}{2}\left(-\frac{1/4}{x+1} + \frac{1/4}{x-3}\right) \\
&= \frac{3}{8}\left(\frac{1}{x-3}\right) + \frac{1}{8}\left(\frac{1}{x+1}\right)
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
\int_0^2 \frac{x}{2x^2-4x-6}\,dx = \frac{3}{8}\int_0^2 \frac{1}{x-3}\,dx + \frac{1}{8}\int_0^2 \frac{1}{x+1}\,dx &= \left[\frac{3}{8}\ln|x-3| + \frac{1}{8}\ln|x+1|\right]_0^2 \\
&= \frac{1}{8}\ln 3 - \frac{3}{8}\ln 3 \\
&= -\frac{\ln 3}{4}
\end{aligned}
$$

(3) Note that

$$
\begin{aligned}
\frac{1}{x^3-2x^2-x+2} \equiv \frac{1}{x^2(x-2)-(x-2)} &= \frac{1}{(x^2-1)(x-2)} \\
&= \frac{1}{(x-1)(x+1)(x-2)}
\end{aligned}
$$

Then, we let

$$
\frac{1}{(x-1)(x+1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2}
$$

where A, B and C are real constants.

Multiplying both sides by $(x-1)(x+1)(x-2)$ and gives

$$1 \equiv A(x+1)(x-2) + B(x-1)(x-2) + C(x-1)(x+1)$$

Putting $x = 1$ on both sides, we have $1 = A(2)(-1) + 0 + 0$ and so $A = -\frac{1}{2}$.

Putting $x = 2$ on both sides, we have $1 = 0 + 0 + C(1)(3)$ and so $C = \frac{1}{3}$.

Putting $x = -1$ on both sides, we have $1 = 0 + B(-2)(-3) + 0$ and so $B = \frac{1}{6}$.

Therefore, we have

$$\frac{1}{(x-1)(x+1)(x-2)} \equiv -\frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{6}\left(\frac{1}{x+1}\right) + \frac{1}{3}\left(\frac{1}{x-2}\right)$$

and thus

$$\begin{aligned}\int \frac{1}{x^3 - 2x^2 - x + 2} dx &= -\frac{1}{2}\int \frac{1}{x-1} dx + \frac{1}{6}\int \frac{1}{x+1} dx + \frac{1}{3}\int \frac{1}{x-2} dx \\ &= -\frac{1}{2}\ln|x-1| + \frac{1}{6}\ln|x+1| + \frac{1}{3}\ln|x-2| + C\end{aligned}$$

where C is an arbitrary constant.

(iv) Note that $\dfrac{1}{e^x - e^{-x}} = \dfrac{1}{2}\left(\dfrac{2}{e^x - e^{-x}}\right) = \dfrac{1}{2\sinh x}$.

Hence, we have

$$\begin{aligned}\int_1^2 \frac{1}{e^x - e^{-x}} dx &= \frac{1}{2}\int_1^2 \frac{1}{\sinh x} \cdot \frac{\sinh x}{\sinh x} dx \\ &= \frac{1}{2}\int_1^2 \frac{d(\cosh x)}{\cosh^2 x - 1} \\ &= \frac{1}{4}\int_1^2 \left(\frac{1}{\cosh x - 1} - \frac{1}{\cosh x + 1}\right) d(\cosh x) \\ &= \frac{1}{4}\left[\ln|\cosh x - 1| - |\cosh x + 1|\right]_1^2 \\ &= \frac{1}{4}\ln\left(\frac{\cosh 2 - 1}{\cosh 2 + 1}\right) - \frac{1}{4}\ln\left(\frac{\cosh 1 - 1}{\cosh 1 + 1}\right) \\ &= \tanh^{-1}\left(\frac{e}{1 + e + e^2}\right)\end{aligned}$$

2 Matrix

(i) By applying elementary row operations, we write

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 5 & 0 & 1 & 0 \\ 4 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-4R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 & 1 & 0 \\ 0 & 2 & -2 & -4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{2}R_3 \to R_3]{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -1 & -2 & 0 & \frac{1}{2} \end{array}\right) \xrightarrow[R_2+R_3 \to R_3]{-R_2+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{8}{3} & \frac{1}{3} & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{8}{3} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & 0 \end{array}\right)$$

Thus, we have $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 5 \\ 4 & 2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5}{3} & -\frac{1}{3} & 0 \\ -\frac{8}{3} & \frac{1}{3} & \frac{1}{2} \\ -\frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}.$

(ii) We apply the elementary row operations as part (i), then

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 9 & 5 & 2 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-2R_1+R_3 \to R_3]{-\frac{9}{2}R_1+R_2 \to R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 5 & -7 & -\frac{9}{2} & 1 & 0 \\ 0 & 3 & -4 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{2}R_1 \to R_1]{\frac{1}{5}R_2 \to R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{7}{5} & -\frac{9}{10} & \frac{1}{5} & 0 \\ 0 & 3 & -4 & -2 & 0 & 1 \end{array}\right) \xrightarrow{-3R_2+R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{7}{5} & -\frac{9}{10} & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{7}{10} & -\frac{3}{5} & 1 \end{array}\right)$$

$$\xrightarrow{5R_3 \to R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{7}{5} & -\frac{9}{10} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{7}{2} & -3 & 5 \end{array}\right) \xrightarrow[\frac{7}{5}R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 3 & -5 \\ 0 & 1 & 0 & 4 & -4 & 7 \\ 0 & 0 & 1 & \frac{7}{2} & -3 & 5 \end{array}\right)$$

Thus, we have $\begin{pmatrix} 2 & 0 & 2 \\ 9 & 5 & 2 \\ 4 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 3 & -5 \\ 4 & -4 & 7 \\ \frac{7}{2} & -3 & 5 \end{pmatrix}$

3 Basis and linear independence

(i) (1) Since $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$, so the collection of vectors form is linearly dependent.

As it is linear dependent set, so it does not form a basis for $\mathbb{R}^3$.

(2) Suppose that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ which is in row reduced echelon form, and there are two non-zero rows.

Thus, the collection of vectors form is linearly independent.

As there are only two linear independent vectors, so it is not a basis for $\mathbb{R}^3$.

(3) Since $\begin{vmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} = 4 \neq 0$, so the collection of vectors form is linearly independent.

As there are three linear independent vectors, so it forms a basis for $\mathbb{R}^3$.

(ii) False. Taking $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (1, 1, 0)$ and $\mathbf{c} = (0, 1, 0)$.

It is clear that $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{c}, \mathbf{a}\}$ are linearly independent sets.

However, since $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$, so $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is not independent set.

Note. You may easily see that $\mathbf{b} = \mathbf{a} + \mathbf{c}$, so clearly $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is not independent set.

(iii) (1) " $\Longrightarrow$ " Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent set, then

$$a\mathbf{u} + b\mathbf{v} = \mathbf{0} \implies a = b = 0$$

Therefore, consider the equation

$$\alpha(\mathbf{u} + \mathbf{v}) + \beta(\mathbf{u} - \mathbf{v}) = \mathbf{0}$$
$$(\alpha + \beta)\mathbf{u} + (\alpha - \beta)\mathbf{v} = \mathbf{0}$$

From the above, we have $\begin{cases} \alpha + \beta = 0 \\ \alpha - \beta = 0 \end{cases}$ and gives $\alpha = \beta = 0$.

By definition, $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

" $\Longleftarrow$ " On the other hand, suppose that

$$a(\mathbf{u} + \mathbf{v}) + b(\mathbf{u} - \mathbf{v}) = \mathbf{0} \implies a = b = 0$$

Therefore, consider the equation

$$\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$$
$$\frac{\alpha+\beta}{2}(\mathbf{u}+\mathbf{v}) + \frac{\alpha-\beta}{2}(\mathbf{u}-\mathbf{v}) = \mathbf{0}$$

From the above, we have $\begin{cases} \frac{\alpha+\beta}{2} = 0 \\ \frac{\alpha-\beta}{2} = 0 \end{cases}$ and gives $\alpha = \beta = 0$.

By definition, $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

(2) I will demonstrate an alternative method to show the linearly independency. Of course, you can proceed similarly as part (1).

"$\Longrightarrow$ Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set, then the matrix

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} \xrightarrow{\text{Row operations}} E$$

where E is the reduced row echlon form.

As $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent, so E has no zero row. Since the matrix

$$\begin{pmatrix} \mathbf{u}+\mathbf{v} \\ \mathbf{v}+\mathbf{w} \\ \mathbf{w}+\mathbf{u} \end{pmatrix} \xrightarrow{-R_1+R_2\to R_2} \begin{pmatrix} \mathbf{u}+\mathbf{v} \\ \mathbf{w}-\mathbf{u} \\ \mathbf{w}+\mathbf{u} \end{pmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{pmatrix} \mathbf{u}+\mathbf{v} \\ \mathbf{w}-\mathbf{v} \\ 2\mathbf{w} \end{pmatrix}$$
$$\xrightarrow[\frac{1}{2}R_3\to R_3]{-\frac{1}{2}R_3+R_2\to R_2} \begin{pmatrix} \mathbf{u}+\mathbf{v} \\ -\mathbf{v} \\ \mathbf{w} \end{pmatrix} \xrightarrow[-R_2\to R_2]{R_2+R_1\to R_1} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} \xrightarrow{\text{Row operations}} E$$

and E has no zero row. Thus, $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$ is linearly independent set.

"$\Longleftarrow$ If $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$ is linearly independent set, then

$$\begin{pmatrix} \mathbf{u}+\mathbf{v} \\ \mathbf{v}+\mathbf{w} \\ \mathbf{w}+\mathbf{u} \end{pmatrix} \xrightarrow{\text{Row operations}} E'$$

where E' is the reduced row echlon form and it has no zero row.

From the above, since all row operations are invertible, so we have

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} \sim \begin{pmatrix} \mathbf{u}+\mathbf{v} \\ \mathbf{v}+\mathbf{w} \\ \mathbf{w}+\mathbf{u} \end{pmatrix} \sim E'$$

and E' has no zero row. Thus, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent set.

4 Linear Transformation

(i) (1) Since $\mathbb{R}^2$ can be spanned by an orthonormal basis $\{(1,0),(0,1)\}$, so it sufficies to consider the reflection on basis vectors to get the matrix representation.

- Step 1: We find a straight line which is perpendicular to $y = ax$ and passes through $(1,0)$.
  For $a > 0$, the equation of straight line is $y = -\dfrac{1}{a}(x-1)$.
- Step 2: We find the point of intersection of the straight line $y = -\dfrac{1}{a}(x-1)$ and the reflection line $y = ax$.
  By solving $\begin{cases} y = ax \\ y = -\dfrac{1}{a}(x-1) \end{cases}$, we get $(x,y) = \left(\dfrac{1}{a^2+1}, \dfrac{a}{a^2+1}\right)$.
- Step 3: Find the coordinates of the reflection point of $(1,0)$ by $y = ax$.
  Note that the point of intersection is the mid-point of $(1,0)$ and the reflection point.
  Denote the reflection point by (p,q), then we have

$$\begin{cases} \dfrac{p+1}{2} & = \dfrac{1}{a^2+1} \\ \dfrac{q+0}{2} & = \dfrac{a}{a^2+1} \end{cases} \implies \begin{cases} p & = \dfrac{1-a^2}{a^2+1} \\ q & = \dfrac{2a}{a^2+1} \end{cases}$$

- Step 4: Denote the matrix representation of the reflection by T, then we have

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{1-a^2}{a^2+1} \\ \dfrac{2a}{a^2+1} \end{pmatrix}$$

- Step 5: Proceed as previous to get $T(0,1)$ as well, we have

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{2a}{a^2+1} \\ \dfrac{a^2-1}{a^2+1} \end{pmatrix}$$

Thus, the required matrix representation is

$$\begin{aligned} T\begin{pmatrix} x \\ y \end{pmatrix} &= \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} \dfrac{1-a^2}{a^2+1} & \dfrac{2a}{a^2+1} \\ \dfrac{2a}{a^2+1} & \dfrac{a^2-1}{a^2+1} \end{pmatrix} \end{aligned}$$

Remark. This matrix is the same as solving $\tan\theta = a$ and plug into the reflection matrix

$$\begin{pmatrix} \cos(2\tan^{-1} a) & \sin(2\tan^{-1} a) \\ \sin(2\tan^{-1} a) & -\cos(2\tan^{-1} a) \end{pmatrix}$$

(2) Recall that the matrix representation of rotating a vectors by an angle θ anti-clockwisely about the z-axis is

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, we want to find a matrix representation of rotating a vector by $30°$ clockwisely about the z-axis, so it flips an orientation of $R_z(\theta)$ and thus the matrix is

$$R_z(-30°) = \begin{pmatrix} \cos 30° & \sin 30° & 0 \\ -\sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(3) Note that we want to find a matrix representation of T so that

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ y \\ z \end{pmatrix}$$

It is not hard to see that the matrix representation required is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(ii) No, since the translation of vectors in $\mathbb{R}^2$ is not linear, it is impossible to find a 2×2 matrix to represent.

5 Others

(i) Without loss of generality, suppose that f attained local maximum in $c \in (a, b)$.

By definition, there exists $\delta > 0$ such that $f(x) \leq f(c)$ for any $x \in (c - \delta, c + \delta)$.

If $f'(c)$ exists, then by the **Carathéodory's Theorem**, there exists a function g defined on (a, b) such that $f(x) - f(c) = g(x)(x - c)$ for any $x \in (a, b)$, and also g is continuous at c with $g(c) = f'(c)$.

It follows that $g(x)(x - c) \leq 0$ for any $x \in (c - \delta, c + \delta)$, that is

$$g(x) \begin{cases} \geq 0 & \text{for } x \in (c - \delta, c) \\ \leq 0 & \text{for } x \in (c, c + \delta) \end{cases}$$

As the function g is continuous at c, so we have $g(c) = 0$, i.e. $f'(c) = 0$.

For f attaining the local minimum in $c \in (a, b)$, it is similar.

Remark. The definition provided in the problem set is called the **Fréchet Derivative**.

(ii) To answer the first question: No. For example, take $f(x) = x$, and $[a, b] = [0, 1]$.

Of course, f attains unique minimum at $x = 0$ and unique maximum at $x = 1$, but $f'(x) = 1 \neq 0$ for any $x \in (0, 1)$.

To answer the second question: No. The converse does not true.

Taking $f(x) = x^3$, and $[a, b] = [-1, 1]$, we have $f'(0) = 0$, and $0 \in (-1, 1)$ but f does not attain any local extremum at $x = 0$.

(iii) Construct a function $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

for any $x \in [a, b]$. Clearly, g is continuous on $[a, b]$ satisfied $g(a) = g(b) = 0$.

Moreover, g' exists for all $x \in (a, b)$ as f is differentiable on (a, b).

Applying the Rolle's theorem, there exists $c \in (a, b)$ such that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

that is $f'(c) = \frac{f(b)-f(a)}{b-a}$ and thus completes the proof.

(iv) Taking arbitrary $p \in (a, b)$. Letting $x \in (p, b]$. By the result of part (iii), there exists $c \in (p, x)$ such that

$$f(x) - f(p) = f'(c)(x - p)$$

If $f'(x) = 0$ for any $x \in (a, b)$, then $f(x) = f(p)$ for all $x \in [p, b]$.

On the other hand, letting $y \in [a, p)$. By the result of part (iii), there exists $c^* \in (a, p)$ such that

$$f(p) - f(y) = f'(c^*)(p - y)$$

Since $f'(x) = 0$ for any $x \in (a, b)$, so $f(y) = f(p)$ for all $y \in [a, p]$.

Combining the above, we have $f(x) = f(y) = f(p)$ for all $y \in [a, p], x \in [p, b]$.

As p is arbitrary, so f is constant on $[a, b]$.

Alternative Solution (Provided by Chow Chung To).

Suppose f is non-constant on $[a, b]$, there exists $m, n \in [a, b]$ with $m \neq n$ such that

$$f(m) \neq f(n).$$

Without loss of generality, assume $m > n$.

By the Mean-Value theorem, there exists $c \in (n, m)$ such that

$$f'(c) = \frac{f(m) - f(n)}{m - n}$$

By the assumption that $f'(x) = 0$ for any $x \in (a, b)$, that implies

$$f'(c) = 0 = \frac{f(m) - f(n)}{m - n}$$

as $c \in (n, m) \subseteq (a, b)$. Contradiction aries and similar when $m < n$.

Thus, f is constant on $[a, b]$.